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Deformation of Lie derivative and μ-symmetries

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Abstract
We introduce, in the spirit of Witten's gauging of exterior differential, a deformed Lie derivative that allows a geometrical interpretation of λ- and μ-symmetries, in complete analogy with standard symmetries. The case of variational symmetries (both for ODEs and for PDEs) is also considered in this approach, leading to λ- and μ-conservation laws.

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1. Introduction

The importance of symmetries in the study of differential equations has been well known since a long time back [4, 8, 15, 25, 26, 28]. In recent years, increasing attention was devoted to the study of λ- and μ-symmetries for ODEs and PDEs respectively; we will not discuss here the relevance of this new class of symmetries, referring to [5, 7, 9, 10, 20–23, 27] for more details and applications. It should be stressed that λ- and μ-symmetries are not symmetries in the proper sense (i.e. they do not map solutions into solutions); nevertheless they can be used to perform symmetry reduction via exactly the same method used for standard symmetries, and they can be interpreted in terms of nonlocal symmetries [5].

The aim of this paper is to give a geometrical interpretation of λ- and μ-symmetries in terms of a deformation of the usual Lie derivative. In particular, following Witten's idea ([29], see also [11, 18]) of gauging the exterior differential operator with a function f, we define the deformed Lie derivative $\mathcal{L}_X^{\mathrm{d}f} = e^f \mathcal{L}_{(e^f X)}$. By using a generalization of this deformed Lie derivative, we recognize that λ- and μ-symmetries can be characterized in complete analogy with standard symmetries by just replacing $\mathcal{L}_X$ with the deformed Lie derivative $\mathcal{L}_X^{\mu}$, where μ is a horizontal 1-form on $J^1(M)$ satisfying suitable conditions (see below).

In particular we show that a vector field on $J^k(M)$ is the λ- or μ-prolongation of a vector field on M if and only if, for any contact form ϑ in $J^k(M)$, $\mathcal{L}_X^{\mu}(\vartheta)$ is a contact form (we recall that the standard prolonged vector fields on $J^k(M)$ can be characterized by requiring that they preserve the contact ideal; see e.g. [2, 15, 26]).

By using the deformed Lie derivative we clarify, from a geometrical point of view, how a λ-symmetry for an ODE can be used to lower the order of the ODE (see [21, 27]). In fact, the definition of a λ-prolonged vector field as a vector field X such that $\mathcal{L}_X^\mu$ preserves the contact ideal allows the construction of a complete system of invariants for X by derivation of lower order invariants. By using this property, given a λ-symmetry X for an nth-order differential equation $\Delta(x, u, u^{(n)}) = 0$ and a complete set of invariants $\{y, w, w_1, \ldots, w_{(n-1)}\}$ for X, the original equation can be written in terms of these invariants as an $(n-1)$ th-order reduced equation $\tilde{\Delta}(y, w, w^{(n-1)}) = 0$.

Finally, we give a definition of λ and μ variational symmetries in terms of the deformed operator $\mathcal{L}_X^\mu$. Given a first-order regular Lagrangian L, we denote by Θ the corresponding Poincaré–Cartan form [3, 12, 16, 17, 26]. By means of Θ we define a variational (λ- or) μ-symmetry as a (λ- or) μ-prolonged vector field X on $J^1(M)$ such that $\mathcal{L}_X^\mu(\Theta)$ is a contact form (standard variational symmetries can be defined as prolonged vector fields X on $J^1(M)$ such that $\mathcal{L}_X(\Theta)$ is a contact form; see the proof of lemma 5 with $\lambda = 0$). In this geometrical framework, we can easily associate with any (λ- or) μ-symmetry X for the variational problem a (λ- or) μ-conservation law (see below and [6, 20] for the exact definition of this concept). Moreover, we prove that our definition of λ and μ variational symmetries completely agrees with the corresponding ones given in local coordinates in [6, 20].

An outline of the paper is as follows. In section 2, in order to fix notations, we collect some basic definitions and preliminary results about jet bundles and differential equations, together with definitions of λ- and μ-symmetries. In section 3 we introduce the deformed operators $\mathcal{L}_X^\mu$ and d^μ, starting from the basic case and considering natural extensions useful for later applications. In section 4 we show how the differential operator introduced in section 3 can be used in order to define λ- and μ-symmetries and to perform reduction of ODEs. Finally, in section 5 we analyze the variational case, both for ODEs and PDEs.

2. Preliminaries

In this section, in order to fix notations and for the convenience of the reader, we collect some definitions and preliminary results about jet bundles and geometry of differential equations (see, e.g., [8, 15, 25, 28]). Moreover we recall the definitions of standard symmetries and λ (or μ)-symmetries, as given in [7, 9, 21].

2.1. Jet bundles and differential equations

Let us consider a fiber bundle (M, π_0, B) in which we introduce local coordinates (x^i, u^a), where $i = 1, \ldots, n$ and $a = 1, \ldots, q$. We will denote by $\Gamma(M)$ the set of local sections of this bundle, by $\mathcal{X}(M)$ the set of vector fields on M and by $\Lambda^*(M)$ the graded algebra of differential forms on M. Finally, $\Lambda^k(M)$ will denote the set of k-forms on M. Similar notations will be used for any fiber bundle. The bundle (M, π_0, B) can be prolonged to the kth *jet bundle* $(J^k(M), \pi_k, B)$ with local coordinates $\left(x^i, u^a, u^a_J\right)$, where J is a multi-index, with $|J| = 1, \ldots, k$. The total space of the jet bundle is also called the jet space, for short.

The jet space $J^k(M)$ is naturally equipped with the canonical *contact forms* that, in local coordinates, read

$$\vartheta^a_J := \mathrm{d}u^a_J - u^a_{J,m}\,\mathrm{d}x^m$$

with $a = 1, \ldots, q, |J| = 0, \ldots, k-1$. We denote by $\mathcal{C}$ the exterior ideal generated by ϑ^a_J, i.e. the set of all the forms in $\Lambda^*(J^k(M))$ that can be written as $\rho^J_a \wedge \vartheta^a_J$ where $\rho^J_a \in \Lambda^*(J^k(M))$

(here and in the following, Einstein's summation convention on repeated multi-indices is assumed). We will call $\mathcal{C}$ the *contact ideal*. We denote by $\mathcal{D}$ the distribution dual to the space of contact 1-forms (see [2, 15]). As is well known, $\mathcal{D}$ is generated by $\partial_{u^a_I}$ with $|I| = k$ and

$$D_i := (\partial/\partial x^i) + u^a_{J,i}\left(\partial\big/\partial u^a_J\right),$$

where $|J| = 1, \ldots, k-1$. The vector field D_i defines a first-order differential operator called *total derivative* with respect to the independent variable x^i.

A 1-form $\mu \in \Lambda^1(J^k(M))$ is called *horizontal* if it annihilates all the vertical tangent directions in $J^k(M)$. Thus, in local coordinates, a horizontal 1-form can be written as $\mu = \Lambda_i\left(x, u^a, u^a_J\right) \mathrm{d}x^i$. With any 1-form ω on $J^k(M)$ is intrinsically associated a horizontal form ω_H on $J^{k+1}(M)$, called the *horizontal component* of ω, which is defined so that $\omega = \omega_H + \vartheta$, where ϑ is a contact form on $J^k(M)$. Given a function $F \in C^\infty(J^k(M))$, we define the *total differential* DF of F as the horizontal component of its ordinary exterior differential $\mathrm{d}F$, i.e. $DF = (\mathrm{d}F)_H$. In terms of the total derivatives of F, we have $DF = (D_i F)\,\mathrm{d}x^i$.

Given a section $\gamma \in \Gamma(M)$, we can consider its *kth order jet extension* $j^k(\gamma) \in \Gamma(J^k(M))$, requiring that $j^k(\gamma)$ coincides with γ on M and annihilates all the contact forms on $J^k(M)$. Let Δ be a differential equation of order k:

$$\Delta := F\left(x^i, u^a, u^a_J\right) = 0,$$

with $|J| = 1, \ldots, k$. If Δ satisfies suitable regularity conditions (see e.g. [1, 2]), we can see Δ as a submanifold of $J^k(M)$: in this case x^i and u^a are independent and dependent variables respectively. A section $\gamma \in \Gamma(M)$ is a *solution* of Δ iff its kth order jet extension satisfies $j^k(\gamma) \subseteq \Delta$.

2.2. Standard and μ-symmetries

Given a vector field $X_0 \in \mathcal{X}(M)$ we define its kth order prolongation as the unique vector field $X \in \chi(J^k(M))$ which reduces to X_0 when restricted to M and which preserves the contact ideal $\mathcal{C}$, i.e. $\mathcal{L}_X(\vartheta) \in \mathcal{C}$ for any $\vartheta \in \mathcal{C}$. If X_0 and X are given in local coordinates by

$$\begin{aligned} X_0 &= \xi^i(x,u)\frac{\partial}{\partial x^i} + \varphi^a(x,u)\frac{\partial}{\partial u^a} \\ X &= X_0 + \Psi^a_J \frac{\partial}{\partial u^a_J}, \end{aligned} \tag{2.1}$$

then the coefficients Ψ^a_J of X satisfy the *prolongation formula*:

$$\Psi^a_{J,k} = D_k \Psi^a_J - u^a_{J,m} D_k \xi^m, \tag{2.2}$$

with $|J| = 0, \ldots, k-1$ and $\Psi^b_0 = \varphi^b$.

Given a kth order differential equation $\Delta := F\left(x^i, u^a, u^a_J\right) = 0$, we say that a vector field $X_0 \in \chi(M)$ is a *symmetry* of Δ iff the kth order prolongation X of X_0 satisfies $X(F) = 0$ on $F = 0$.

In recent years a new class of symmetries, christened λ (or μ)-symmetries, for a differential equation was introduced (see [9, 21]), sharing the useful properties of standard symmetries for what concern reduction of ODEs and determination of invariant solutions for PDEs. The notion of λ- and μ-symmetries is based on a different choice of the prolongation formula. In particular, for a single ODE, Muriel and Romero [21] define the λ-*prolongation* of a vector field X_0 as a vector field $X \in \chi(J^k(M))$ of the form (2.1) whose coefficients satisfy the following prolongation formula, depending on a smooth function $\lambda \in C^\infty(J^1(M))$:

$$\Psi_{n+1} = (D_x + \lambda)\Psi_n - u_{n+1}(D_x + \lambda)\xi \tag{2.3}$$

(here we simply write u_n for $D_x^n u$ and similarly for Ψ_n). This formula was generalized to the case of PDEs in [7, 9]. In particular, if (M, π_0, B) is a fiber bundle equipped with a horizontal 1-form $\mu = \Lambda_i \mathrm{d}x^i$, we can define the kth order μ-prolongation of a vector field $X_0 \in \chi(M)$ as the unique vector field $X \in \chi(J^k(M))$ of the form (2.1) whose coefficients satisfy the μ-prolongation formula:

$$\Psi^a_{J,k} = [D_k + \Lambda_k]\,\Psi^a_J - u^a_{J,m}\,[D_k + \Lambda_k]\,\xi^m. \tag{2.4}$$

Here, $\Psi^b_0 = \varphi^b$ and Λ_i satisfy the compatibility conditions

$$D_i \Lambda_j - D_j \Lambda_i = 0. \tag{2.5}$$

In terms of the horizontal 1-form μ, the compatibility conditions (2.5) can be written as $\mathrm{d}\mu \in \mathcal{C}$.

Note that if we consider $\mu \in \Lambda^1(J^1(M))$, we can guarantee that the μ-prolongation of a Lie-point vector field on M is a proper vector field in each $J^k(M)$. On the other side, if we consider $\mu \in \Lambda^1(J^r(M))$ with $r > 1$, the μ-prolongation of X_0 would be a generalized vector field in each $J^k(M)$ with $k > 0$ even if X_0 is a Lie-point vector field.

Given a kth order differential equation $\Delta := F\left(x^i, u^a, u^a_J\right) = 0$, we say that a vector field $X_0 \in \chi(M)$ is a λ-*symmetry* (or a μ-*symmetry* in the case of PDEs) of Δ iff the kth order λ (or μ)-prolongation X of X_0 satisfies $X(F) = 0$ on $F = 0$.

3. The deformed operators d^μ and $\mathcal{L}^\mu$

Deformed differential operators appeared in many parts of the geometric theory of differential equations (see, e.g., [11, 18, 19, 29]). In this section, we define a deformation of the differential operators d and $\mathcal{L}$ in the spirit of Witten's gauging of the exterior derivative ([29], see also [11, 18]).

3.1. The basic operators

We start by defining the deformed differential operators d^μ and $\mathcal{L}^\mu$ in the simplest case, where we gauge the standard operators d and $\mathcal{L}$ by means of a function $f \in C^\infty(M)$.

Definition 1. *Let M be a differential manifold and $f \in C^\infty(M)$. For any $\beta \in \Lambda^*(M)$, we define the deformed differential $\mathrm{d}^{\mathrm{d}f}$ by*

$$\mathrm{d}^{\mathrm{d}f}\beta := e^{-f}\mathrm{d}(e^f\beta) = \mathrm{d}\beta + \mathrm{d}f \wedge \beta.$$

Lemma 1. *The deformed exterior differential $\mathrm{d}^{\mathrm{d}f}$ is a first-order differential operator with respect to the wedge product on differential forms, satisfying*

$$\mathrm{d}^{\mathrm{d}f} \circ \mathrm{d}^{\mathrm{d}f} = 0.$$

Proof. The proof is a straightforward computation based on the fact that $\mathrm{d}(\mathrm{d}f) = 0$ and $\mathrm{d}f \wedge \mathrm{d}f = 0$. □

Following the same line of reasoning, we give the definition of the deformed Lie derivative of a form $\beta \in \Lambda^*(M)$ along a vector field $X \in \chi(M)$ as follows.

Definition 2. *Let M be a differential manifold, $f \in C^\infty(M)$, $\beta \in \Lambda^*(M)$ and $X \in \chi(M)$. Then we define the deformed Lie derivative $\mathcal{L}^{\mathrm{d}f}_X$ by*

$$\mathcal{L}^{\mathrm{d}f}_X\beta := e^{-f}\mathcal{L}_{(e^f X)}\beta = \mathcal{L}_X\beta + \mathrm{d}f \wedge (X \lrcorner \beta).$$

Lemma 2. *The deformed Lie derivative* $\mathcal{L}_X^{\mathrm{d}f}$ *satisfies*

$$\mathcal{L}_X^{\mathrm{d}f}(\beta_1 + \beta_2) = \mathcal{L}_X^{\mathrm{d}f}\beta_1 + \mathcal{L}_X^{\mathrm{d}f}\beta_2$$
$$\mathcal{L}_X^{\mathrm{d}f}(\beta_1 \wedge \beta_2) = \left(\mathcal{L}_X^{\mathrm{d}f}\beta_1\right) \wedge \beta_2 + \beta_1 \wedge \left(\mathcal{L}_X^{\mathrm{d}f}\beta_2\right)$$

for any $X \in \chi(M)$ *and* $\beta_1, \beta_2 \in \Lambda^*(M)$. *Moreover,* $\mathcal{L}_X^{\mathrm{d}f}$ *coincides with the usual Lie derivative on functions.*

Proof. The proof is an easy computation. □

It is now completely natural to extend definition 2 to the case of Lie derivative of vector fields by the following.

Definition 3. *Let M be a differential manifold,* $f \in C^\infty(M)$, *and* $X, Y \in \chi(M)$. *Then*

$$\mathcal{L}_X^{\mathrm{d}f}(Y) := e^{-f}\mathcal{L}_{(e^f X)}Y = \mathcal{L}_X Y - (Y \lrcorner \mathrm{d}f)X.$$

Lemma 3. *The deformed Lie derivative* $\mathcal{L}_X^{\mathrm{d}f}$ *satisfies*

$$\mathcal{L}_X^{\mathrm{d}f}(Y \lrcorner \beta) = \mathcal{L}_X^{\mathrm{d}f}(Y) \lrcorner \beta + Y \lrcorner \left(\mathcal{L}_X^{\mathrm{d}f}\beta\right)$$

for any $X, Y \in \chi(M)$ *and* $\beta \in \Lambda^*(M)$.

Proof. The proof is a straightforward computation and is left to the reader. □

The aim of the following theorem is to give an analogous of Cartan formula for the deformed Lie derivative $\mathcal{L}_X^{\mathrm{d}f}$ given by definition 2.

Theorem 1. *Let M be a differential manifold,* $f \in C^\infty(M)$, $X \in \chi(M)$ *and* $\beta \in \Lambda^*(M)$. *Then*

$$(a)\ \mathcal{L}_X^{\mathrm{d}f}(\beta) = X \lrcorner \mathrm{d}\beta + \mathrm{d}^{\mathrm{d}f}(X \lrcorner \beta)$$
$$(b)\ \mathcal{L}_X^{\mathrm{d}f}(\mathrm{d}\beta) = \mathrm{d}^{\mathrm{d}f}\mathcal{L}_X^{\mathrm{d}f}(\beta).$$

Proof. The point (*a*) follows directly from definitions 1 and 2. In order to prove (*b*) we apply (*a*) to the form $\mathrm{d}\beta$ and we find $\mathcal{L}_X^{\mathrm{d}f}(\mathrm{d}\beta) = X \lrcorner \mathrm{d}(\mathrm{d}\beta) + \mathrm{d}^{\mathrm{d}f}(X \lrcorner \mathrm{d}\beta) = \mathrm{d}^{\mathrm{d}f}(X \lrcorner \mathrm{d}\beta)$. On the other hand, by lemma 1 and applying $\mathrm{d}^{\mathrm{d}f}$ to (*a*) we find $\mathrm{d}^{\mathrm{d}f}\left(\mathcal{L}_X^{\mathrm{d}f}\beta\right) = \mathrm{d}^{\mathrm{d}f}(X \lrcorner \mathrm{d}\beta)$; this proves (*b*). □

3.2. Generalizations

In the previous subsection, we started with a function $f \in C^\infty(M)$ in order to define the deformed differential operators $\mathrm{d}^{\mathrm{d}f}$ and $\mathcal{L}_X^{\mathrm{d}f}$. On the other hand, it is immediate to recognize that these differential operators depend on $\mathrm{d}f$ rather than on f. So, into all previous definitions, we can substitute $\mathrm{d}f$ with a closed 1-form $\mu \in \Lambda^1(M)$. Let us make this extension more explicit by the following.

Definition 4. *Let M be a differential manifold and* μ *a closed 1-form on M. Then, for any* $\beta \in \Lambda^*(M)$ *and* $X, Y \in \chi(M)$, *we define*

$$\mathrm{d}^\mu\beta := \mathrm{d}\beta + \mu \wedge \beta$$
$$\mathcal{L}_X^\mu\beta := \mathcal{L}_X\beta + \mu \wedge (X \lrcorner \beta)$$
$$\mathcal{L}_X^\mu(Y) := \mathcal{L}_X Y - (Y \lrcorner \mu)X.$$

Theorem 2. *Let M be a differential manifold and $\mu \in \Lambda^1(M)$. Then for any $X \in \chi(M)$ and $\beta \in \Lambda^*(M)$, the following conditions are equivalents:*

$$(a)\ \mathrm{d}\mu = 0$$
$$(b)\ \mathrm{d}^\mu \circ \mathrm{d}^\mu = 0$$
$$(c)\ \mathcal{L}_X^\mu \,\mathrm{d}\beta = \mathrm{d}^\mu\big(\mathcal{L}_X^\mu \beta\big).$$

Proof. By explicit computation, we find

$$\mathrm{d}^\mu(\mathrm{d}^\mu(\beta)) = \mathrm{d}^\mu(\mathrm{d}\beta + \mu \wedge \beta) = \mathrm{d}\mu \wedge \beta = 0$$
$$\mathcal{L}_X^\mu \,\mathrm{d}\beta = \mathrm{d}^\mu(X \lrcorner \,\mathrm{d}\beta) = \mathrm{d}^\mu\big(\mathcal{L}_X^\mu \beta\big) - \mathrm{d}^\mu(\mathrm{d}^\mu(X \lrcorner \,\beta));$$

then the equivalence of (a)–(c) follows as X and β are arbitrary. □

3.3. The case of jet bundles

In order to apply our deformed differential operators to the study of symmetries of differential equations (and in particular to λ- and μ-symmetries), we have to adapt the operators d^μ and $\mathcal{L}_X^\mu$ to the framework of jet bundles. In particular, we require that the deformed operator d^μ defines a horizontal cohomology (see [13, 15, 26] and remark 2) and we get a weaker version of theorem 2.

Let $\mu \in \Lambda^1(J^1(M))$ be a horizontal 1-form such that $\mathrm{d}\mu \in \mathcal{C}$ (the introduction of μ is motivated by the μ-prolongation formula (2.4)). Given a local coordinate system $\big(x^i, u^a, u^a_i\big)$ on $J^1(M)$, we can write $\mu = \Lambda_i\big(x^k, u^a, u^a_k\big)\,\mathrm{d}x^i$ with Λ_i real functions satisfying the condition $D_k\Lambda_i = D_i\Lambda_k$. Analogous to the previous cases, we can give the following.

Definition 5. *Let (M, π_0, B) be a fiber bundle and let $\mu \in \Lambda^1(J^1(M))$ be a horizontal 1-form such that $\mathrm{d}\mu \in \mathcal{C}$. Then, for any $\beta \in \Lambda^*(J^k(M))$ and $X, Y \in \chi(J^k(M))$, we define*

$$\mathrm{d}^\mu\beta := \mathrm{d}\beta + \mu \wedge \beta$$
$$\mathcal{L}_X^\mu\beta := \mathcal{L}_X\beta + \mu \wedge (X \lrcorner \,\beta)$$
$$\mathcal{L}_X^\mu(Y) := \mathcal{L}_X Y - (Y \lrcorner \,\mu)X.$$

Note that in this case we cannot guarantee that $\mathrm{d}^\mu \circ \mathrm{d}^\mu = 0$, because we only require that $\mathrm{d}\mu \in \mathcal{C}$. Then we can only prove a weaker version of theorem 2, given by the following.

Theorem 3. *Let (M, π_0, B) be a fiber bundle and let $\mu \in \Lambda^1(J^1(M))$ be a horizontal 1-form. Then, for any $\beta \in \Lambda^*(J^k(M))$ and $X \in \chi(M)$, the following conditions are equivalent:*

$$\begin{aligned}&(a)\ \mathrm{d}\mu \in \mathcal{C}\\ &(b)\ \mathrm{d}^\mu(\mathrm{d}^\mu(\beta)) \in \mathcal{C}\\ &(c)\ \mathcal{L}_X^\mu(\mathrm{d}\beta) - \mathrm{d}^\mu\mathcal{L}_X^\mu(\beta) \in \mathcal{C}.\end{aligned} \tag{3.1}$$

Proof. We proceed by direct computation from definition 5 and we find

$$\mathrm{d}^\mu(\mathrm{d}^\mu\beta) = \mathrm{d}^\mu(\mathrm{d}\beta + \mu \wedge \beta) = (\mathrm{d}\mu + \mu \wedge \mu) \wedge \beta.$$

Then, as $\mu \wedge \mu = 0$ and β is an arbitrary k-form, this proves the equivalence of (a) and (b). Moreover, to prove the equivalence of (b) and (c) we start again by definition 5 and write $\mathcal{L}_X^\mu(\beta) = X \lrcorner \,\mathrm{d}\beta + \mathrm{d}^\mu(X \lrcorner \,\beta)$. Then we have

$$\mathcal{L}_X^\mu(\mathrm{d}\beta) = X \lrcorner \,\mathrm{d}\,\mathrm{d}\beta + \mathrm{d}^\mu(X \lrcorner \,\mathrm{d}\beta) = \mathrm{d}^\mu(X \lrcorner \,\mathrm{d}\beta)$$

and, by considering the d^μ differential of $\mathcal{L}_X^\mu(\beta)$, we find

$$\mathrm{d}^\mu\left(\mathcal{L}_X^\mu(\beta)\right) = \mathrm{d}^\mu(X \lrcorner \mathrm{d}\beta) + \mathrm{d}^\mu(\mathrm{d}^\mu(X \lrcorner \beta).$$

Then $\mathcal{L}_X^\mu(\mathrm{d}\beta) = \mathrm{d}^\mu\left(\mathcal{L}_X^\mu\beta\right) - \mathrm{d}^\mu(\mathrm{d}^\mu(X \lrcorner \beta))$, and the thesis follows as X and β are arbitrary. □

Remark 1. The choice made in definitions 2, 3 and 5 for the deformed Lie derivative $\mathcal{L}_X^\mu$ is motivated by the fact that there is a natural one-to-one correspondence between λ-symmetries and a particular class of nonlocal symmetries [7, 21]. More explicitly, a vector field X is a λ-symmetry of Δ if and only if the vector field $Y = e^{\int\mu}X$ is a (possibly) nonlocal symmetry of Δ (here $\mu = \lambda\,\mathrm{d}x$ and $D_x(\int\mu) = \lambda$). This suggests the idea of considering the deformed Lie derivative along a λ-symmetry as the standard Lie derivative along the corresponding nonlocal symmetry. If we just do it, we find $\mathcal{L}_Y\beta = \mathcal{L}_{(e^{\int\mu}X)}\beta = e^{\int\mu}\mathcal{L}_X\beta + e^{\int\mu}\mu\wedge(X \lrcorner \beta) + \vartheta$, where ϑ is a suitable contact form. Then $\mathcal{L}_Y\beta$ still depends on the (possibly) nonlocal function $e^{\int\mu}$, and the introduction of the factor $e^{-\int\mu}$ allows us to avoid any nonlocality in the definition of $\mathcal{L}_X^\mu$.

Remark 2. If we consider the variational bi-complex obtained by a decomposition of the de Rham complex in terms of the contact ideal $\mathcal{C}$ (see, e.g., [1, 13, 15, 25]), the exterior differential d splits into two pieces $\mathrm{d} = \mathrm{d}_H + \mathrm{d}_V$, where d_H and d_V are anti-commuting co-boundary operators. In particular, the local coordinate expression for d_H is given by

$$\mathrm{d}_H(f) = D_i(f)\,\mathrm{d}x^i, \qquad \mathrm{d}_H(\mathrm{d}x^i) = 0, \qquad \mathrm{d}_H\left(\vartheta_J^a\right) = -\vartheta_{J,i}^a \wedge \mathrm{d}x^i.$$

By means of the horizontal exterior differential d_H, we can associate with d^μ a cohomological operator defined by

$$\mathrm{d}_H^\mu\beta := \mathrm{d}_H\beta + \mu\wedge\beta \quad \forall\beta\in\Lambda^*(J^k(M)),$$

satisfying $\mathrm{d}_H^\mu\left(\mathrm{d}_H^\mu(\beta)\right) = 0\,\forall\beta\in\Lambda^*(J^k(M))$. In this sense, we say that d^μ defines a horizontal cohomology.

4. Application to λ- and μ-symmetries

In this section, we show how λ- and μ-symmetries are the exact analogous of standard symmetries if we substitute the usual Lie derivative with the deformed one.

4.1. Relations between $\mathcal{L}_X^\mu$ and λ- or μ-prolongation

We start by showing how the notion of λ- and μ-prolonged vector fields can be expressed in terms of the deformed Lie derivative. In particular, we will prove that the λ- and μ-prolonged vector fields on $J^k(M)$ can be characterized as a vector field X on $J^k(M)$ such that $\mathcal{L}_X^\mu$ preserves the contact ideal on $J^k(M)$.

Theorem 4. *Let (M, π_0, B) be a fiber bundle and let $\mu\in\Lambda^1(J^1(M))$ be a horizontal 1-form such that $\mathrm{d}\mu\in\mathcal{C}$. Then a vector field $X\in\chi(J^k(M))$ is the λ- or μ-prolongation of a vector field $X_0\in\chi(M)$ if and only if X reduces to X_0 when restricted to M and*

$$\mathcal{L}_X^\mu(\vartheta)\in\mathcal{C} \qquad \forall\,\vartheta\in\mathcal{C}.$$

Proof. If we consider the generators of the contact ideal $\mathcal{C}$ on $J^k(M)$ given in local coordinates by $\vartheta_J^a = \mathrm{d}u_J^a - u_{J,i}^a\,\mathrm{d}x^i$ and the vector field $X\in\chi(J^k(M))$ of the form

$$X = \xi^i\frac{\partial}{\partial x^i} + \varphi^a\frac{\partial}{\partial u^a} + \Psi_J^a\frac{\partial}{\partial u_J^a},$$

a straightforward computation shows that the condition

$$\mathcal{L}_X^{\mu}\big(\vartheta_J^a\big) = \mathcal{L}_X\big(\vartheta_J^a\big) + \big(X \lrcorner\, \vartheta_J^a\big)\mu \in \mathcal{C}$$

leads to the λ-prolongation formula (2.3) (see also [7, 9, 21]) when the base manifold B is one dimensional and to the μ-prolongation formula (2.4) when $\dim(B) > 1$. □

The deformed Lie derivative can also be used to give the following alternative characterization of λ- and μ-prolonged vector fields.

Theorem 5. *Let (M, π_0, B) be a fiber bundle and let $\mu \in \Lambda^1(J^1(M))$ be a horizontal 1-form such that $\mathrm{d}\mu \in \mathcal{C}$. If $\mathcal{D}$ denotes the distribution generated by D_i, then a vector field $X \in \chi(J^k(M))$ is the λ- or μ-prolongation of a vector field $X_0 \in \chi(M)$ if and only if X reduces to X_0 when restricted to M and*

$$\mathcal{L}_X^{\mu}(Y) \in \mathcal{D} \qquad \forall\, Y \in \mathcal{D}.$$

Proof. We just recall that $\mathcal{L}_X^{\mu}(D_i) = \mathcal{L}_X(D_i) - (D_i \lrcorner\, \mu)X = \mathcal{L}_X(D_i) - \Lambda_i X$ and refer to [9] (lemmas 2 and 4) for explicit computation. □

Remark 3. Given an equation $\Delta := F = 0$ in $J^k(M)$ and a λ-symmetry of Δ, the deformed Lie derivative along X is related (but not equivalent) to the standard Lie derivative along the corresponding nonlocal symmetry in a suitable covering of Δ. In fact, the main result of [5] guarantees that a λ-symmetry X for an equation $F = 0$ can be associated with a particular nonlocal symmetry of the form

$$Y = e^w X + C\frac{\partial}{\partial w}$$

in the covering $\{F = 0, w_x = \lambda\}$ of the equation $F = 0$ (for a suitable function C in the covering space). In this framework, we have the following relation between the standard Lie derivative along Y in the covering space and the deformed Lie derivative along X in $J^k(M)$:

$$\mathcal{L}_Y\beta = e^w\mathcal{L}_X^{\mu}(\beta) + \mathcal{L}_{C\partial/\partial_w}\beta.$$

4.2. λ-symmetries and reduction for ODEs

We now want to stress the role of the operator $\mathcal{L}_X^{\mu}$ in the reduction of ODEs. We start by briefly recalling what happens in the case of standard prolonged vector fields in the language of total differentials. Let (M, π_0, B) be a fiber bundle with a one-dimensional base, and let X be a vector field on $J^k(M)$ which is the kth order prolongation of a vector field on M. If $F \in C^{\infty}(J^k(M))$ is a differential invariant for X (i.e. $X(F) = 0$), then $\mathcal{L}_X(DF) \in \mathcal{C}$, where DF denotes the total differential of F. In fact, by definition, $DF = \mathrm{d}F + \vartheta$, where ϑ is a suitable contact form. Then

$$\mathcal{L}_X(DF) = \mathcal{L}_X(\mathrm{d}F + \vartheta) = \mathrm{d}(\mathcal{L}_X(F)) + \mathcal{L}_X(\vartheta) \in \mathcal{C}$$

because $\mathcal{L}_X\mathrm{d} = \mathrm{d}\mathcal{L}_X$, $\mathcal{L}_X(F) = 0$ and $\mathcal{L}_X(\vartheta) \in \mathcal{C}$ since X is a prolonged vector field.

Given two differential invariants F_1 and F_2 (with $D_x(F_i) \neq 0$) for a prolonged vector field X, the corresponding total differentials are necessarily proportional (via a function H) as the base manifold B of the fiber bundle is one dimensional; so we have $DF_1 = H(DF_2)$, where $H = D_xF_1/D_xF_2$. If we consider

$$\mathcal{L}_X(DF_1) = \mathcal{L}_X[H(DF_2)] = \mathcal{L}_X(H)(DF_2) + H\mathcal{L}_X(DF_2),$$

using $\mathcal{L}_X(DF_1) \in \mathcal{C}$ and $\mathcal{L}_X(DF_2) \in \mathcal{C}$, we find $\mathcal{L}_X(H) = X(H) = 0$.

By just considering $\mathcal{L}_X^{\mu}$ instead of $\mathcal{L}_X$, we can prove the following.

Lemma 4. *Let (M, π_0, B) be a fiber bundle with a one-dimensional base, $\mu \in \Lambda^1(J^1(M))$ a horizontal 1-form and $X \in \chi(J^k(M))$ a kth order λ-prolonged vector field. If F is a differential invariant for X, then*

$$\mathcal{L}_X^\mu(DF) \in \mathcal{C}.$$

Proof. Following the steps recalled above for the standard case, we write $DF = \mathrm{d}F + \vartheta$, where ϑ is a suitable contact form. Then

$$\mathcal{L}_X^\mu(DF) = \mathcal{L}_X^\mu(\mathrm{d}F + \vartheta) \in \mathcal{C}$$

if and only if

$$\mathrm{d}^\mu\big(\mathcal{L}_X^\mu(F)\big) + \mathcal{L}_X^\mu(\vartheta) \in \mathcal{C},$$

where we used $\mathcal{L}_X^\mu \mathrm{d} - \mathrm{d}^\mu \mathcal{L}_X^\mu \in \mathcal{C}$ (see theorem 3). Now, recalling that $\mathcal{L}_X^\mu(F) = X(F) = 0$ by hypothesis, and using theorem 4, we have the thesis. Note that in this case the condition $\mathrm{d}\mu \in \mathcal{C}$ is automatically satisfied, as the base manifold B is one dimensional. □

Now we can prove the following theorem, showing that for a λ-prolonged vector field we can construct a complete system of invariants by the derivation of lower order invariants.

Theorem 6. *Let (M, π_0, B) be a fiber bundle with a one-dimensional base manifold, $\mu \in \Lambda^1(J^1(M))$ a horizontal 1-form and $X \in \chi(J^k(M))$ a kth order λ-prolonged vector field. If F_1 and F_2 are differential invariants for X such that $D_x(F_i) \neq 0$, then $D_x F_1/D_x F_2$ is a differential invariant for X.*

Proof. As before, if we consider the total differentials DF_1 and DF_2, they are necessarily proportional 1-forms (as the base manifold B of the fiber bundle is one dimensional), so we have $DF_1 = H(DF_2)$ where $H = D_x F_1/D_x F_2$. Then

$$\mathcal{L}_X^\mu(DF_1) = \mathcal{L}_X^\mu[H(DF_2)] = \mathcal{L}_X^\mu(H)(DF_2) + H\mathcal{L}_X^\mu(DF_2)$$

and, as lemma 4 guarantees $\mathcal{L}_X^\mu(DF_1) \in \mathcal{C}$ and $\mathcal{L}_X^\mu(DF_2) \in \mathcal{C}$, we find $\mathcal{L}_X^\mu(H) = X(H) = 0$. □

Remark 4. The previous results can be extended to fiber bundles with an n-dimensional base manifold B. In this case, we have to consider a horizontal 1-form μ satisfying $\mathrm{d}\mu \in C$. Then, given a kth order μ-prolonged vector field X on $J^k(M)$ and a set of $(n+1)$ differential invariants $F_1, \ldots, F_n, F_{n+1}$ of X, we can construct n new invariants. In particular, if $F_1, \ldots, F_n$ are independent invariant functions, we consider DF_i as a contact-invariant base for the 1-forms on M (see [26]). Then $DF_{n+1} = \sum_{i=1}^n H_i(DF_i)$, and using the same technique as before we can prove that $X(H_i) = 0$. Unfortunately, as explained in [26], this is not sufficient to provide reduction for PDEs.

5. Variational λ- and μ-symmetries

The relation between standard symmetries of the Lagrangian and conservation laws is described by the classical Noether theorem (see, e.g., [3, 12, 14, 24–26]). In this section, we consider variational λ- and μ-symmetries for a first-order regular Lagrangian. In particular we give a geometrical interpretation of the definitions of λ and μ variational symmetries given in [6, 20], and we find the corresponding conservation laws by means of this geometrical framework. All results for ODEs can be extended to the case of higher order Lagrangians by just considering the corresponding Poincaré–Cartan form (see e.g. [17]). In the case of PDE some problem could arise if we consider Lagrangians of order greater than 2, as in this case it is not possible to define the corresponding Poincarè–Cartan n-form in a unique invariant way starting only from the knowledge of the Lagrangian (see [16] and references therein).

5.1. The case of ODEs

Given a fiber bundle (M, π_0, B) with a one-dimensional base, let $J^1(M)$ be the corresponding first-order jet bundle, in which we introduce local coordinates $\left(x, u^a, u^a_x\right)$. If $L : J^1(M) \to \mathbf{R}$ is a first-order regular Lagrangian, we denote by

$$\Theta = \frac{\partial L}{\partial u^a_x}\vartheta^a + L\,\mathrm{d}x \tag{5.1}$$

the Poincaré–Cartan 1-form associated with L (as usual $\vartheta^a = \mathrm{d}u^a - u^a_x\,\mathrm{d}x$ are the contact forms on $J^1(M)$).

Definition 6. *Let $\mu = \lambda\left(x, u^a, u^a_x\right)\mathrm{d}x$ be a horizontal 1-form on $J^1(M)$. A vector field $X_0 \in \chi(M)$ is a variational λ-symmetry for the Lagrangian L iff its first-order λ-prolongation $X \in \chi(J^1(M))$ satisfies*

$$\mathcal{L}^\mu_X(\Theta) \in \mathcal{C}.$$

A vector field $X_0 \in \chi(M)$ is a *divergence variational λ-symmetry* for the Lagrangian L iff its first-order λ-prolongation $X \in \chi(J^1(M))$ satisfies

$$\mathcal{L}^\mu_X(\Theta) - \mathrm{d}^\mu R \in \mathcal{C} \tag{5.2}$$

for a suitable function $R \in C^\infty(M)$.

Lemma 5. *Condition (5.2) of definition 6 is equivalent to*

$$X(L) + L(D_x + \lambda)\xi = (D_x + \lambda)(R),$$

where X is the λ-prolongation of a vector field X_0 on M.

Proof. Writing explicitly condition (5.2) for the vector field $X = \xi\partial_x + \varphi^a\partial_{u^a} + \Psi^a\partial_{u^a_x}$ and the Poincaré–Cartan form Θ given by (5.1), we find

$$\mathcal{L}^\mu_X(\Theta) - \mathrm{d}^\mu R = \left[X\left(\frac{\partial L}{\partial u^a_x}\right) - \frac{\partial L}{\partial u^a}\xi\right]\vartheta^a - \left(\varphi^a - \xi u^a_x\right)\left[\mathrm{d}\left(\frac{\partial L}{\partial u^a_x}\right) - \frac{\partial L}{\partial u^a}\mathrm{d}x\right]$$
$$+\,\mathrm{d}\left[\frac{\partial L}{\partial u^a_x}\left(\varphi^a - \xi u^a_x\right) + L\xi\right] + \left[\frac{\partial L}{\partial u^a_x}\left(\varphi^a - \xi u^a_x\right) + L\xi\right]\lambda\,\mathrm{d}x - \mathrm{d}R - \lambda R\,\mathrm{d}x \in \mathcal{C}.$$

So, we have

$$\left(\xi\frac{\partial}{\partial x} + \varphi^a\frac{\partial}{\partial u^a} + \Psi^a\frac{\partial}{\partial u^a_x}\right)(L) + L(D_x + \lambda)(\xi) = (D_x + \lambda)(R),$$

where $\Psi^a := (D_x + \lambda)\varphi^a - u^a_x(D_x + \lambda)(\xi)$, and we get the thesis by just recalling the λ-prolongation formula (2.3) for the vector field $X_0 = \xi\partial_x + \varphi^a\partial_{u^a}$. □

Finally, we want to show that this characterization of (divergence) variational λ-symmetries leads to an analogous of Noether's theorem. We recall that in this geometrical framework, a conservation law for an ODE Δ is a function P such that $\mathrm{d}P \in \mathcal{C}$ on Δ [3, 15]. If we consider a system of Euler–Lagrange equations, the previous condition is equivalent to requiring that $D_x P = 0$ on the solutions of the Euler–Lagrange equations. We extend this definition of conservation law to the case of deformed operators by the following.

Definition 7. *A λ-conservation law for a given ODE $\Delta := F\left(x, u^a, u^a_n\right) = 0$ is a function P such that $\mathrm{d}^\mu P \in \mathcal{C}$ on $F = 0$. In particular this means that $D_x \lrcorner\, \mathrm{d}^\mu P = D_x(P) + \lambda P = 0$ on $F = 0$.*

Now we can prove the analogous of Noether's theorem for λ-symmetries: when expressed in local coordinates, our result completely agrees with the conservation law found in [20].

Theorem 7. *Let X_0 be a divergence variational λ-symmetry for a first-order regular Lagrangian L. If Θ denotes the Poincaré–Cartan 1-form associated with L, we have the λ-conservation law*

$$D_x(X_0 \lrcorner \Theta - R) + \lambda(X_0 \lrcorner \Theta - R) = 0. \tag{5.3}$$

Proof. We recall that X_0 is a divergence variational λ-symmetry iff its first-order λ-prolongation X satisfies

$$\mathcal{L}_X^\mu \Theta - \mathrm{d}^\mu R = X \lrcorner \mathrm{d}\Theta + \mathrm{d}^\mu (X \lrcorner \Theta - R) \in \mathcal{C} \tag{5.4}$$

for a suitable function $R \in C^\infty(M)$. Recalling that on the solutions to the Euler-Lagrange equations we have $X \lrcorner \mathrm{d}\Theta = 0$, we find (5.3) by just recalling that the form of Θ guarantees $X \lrcorner \Theta = X_0 \lrcorner \Theta$. □

5.2. The case of PDEs

The geometrical framework presented for ODEs in the previous subsection still holds in the PDEs' context, by just considering a suitable Poincaré–Cartan n-form that we still denote by Θ. In particular, given a fiber bundle (M, π_0, B) with an n-dimensional orientable base manifold B, we consider the corresponding first-order jet bundle $J^1(M)$ with local coordinates $\left(x^i, u^a, u_i^a\right)$. If $L = L\left(x^i, u^a, u_i^a\right)$ is a regular first-order Lagrangian, the corresponding Poincaré–Cartan n-form is

$$\Theta = \frac{\partial L}{\partial u_i^a} \vartheta^a \wedge \Omega_i + L\Omega, \tag{5.5}$$

where $\vartheta^a = \mathrm{d}u^a - u_i^a \,\mathrm{d}x^i$ are the contact forms on $J^1(M)$, $\Omega = \mathrm{d}x^1 \wedge \ldots \mathrm{d}x^n$ is the volume form on the base manifold B and $\Omega_i = \partial_i \lrcorner \Omega$.

Now we can define a variational μ-symmetry by the following.

Definition 8. *Let $\mu = \Lambda_i\left(x^k, u^a, u_k^a\right) \mathrm{d}x^i$ be a horizontal 1-form on $J^1(M)$ satisfying $\mathrm{d}\mu \in \mathcal{C}$ and Θ the Poincaré–Cartan form associated with L. A vector field $X_0 \in \chi(M)$ is a variational μ-symmetry for the Lagrangian L iff its first-order μ-prolongation $X \in \chi(J^1(M))$ satisfies*

$$\mathcal{L}_X^\mu(\Theta) \in C. \tag{5.6}$$

A vector field $X_0 \in \chi(M)$ is a *divergence variational μ-symmetry* for the Lagrangian L iff its first-order μ-prolongation $X \in \chi(J^1(M))$ satisfies

$$\mathcal{L}_X^\mu(\Theta) - \mathrm{d}^\mu \rho \in C \tag{5.7}$$

for a suitable $(n-1)$-form ρ. If we write $\rho = R^i \Omega_i + \sigma$ with $\sigma \in \mathcal{C}$, it is an easy computation (see lemma 5) to prove that condition (5.7) can be written, in local coordinates, as

$$X(L) + L(D_i + \Lambda_i)\xi^i = (D_i + \Lambda_i)R^i.$$

Finally we want to show that also in the case of variational PDEs, we can associate with any variational μ-symmetries a μ-conservation law. Let us recall that a conservation law for a PDE Δ can be defined as a $(n-1)$-form π such that $\mathrm{d}\pi \in \mathcal{C}$ on Δ [3, 15]. Generalizing this idea to the case of deformed differential operators, we can give the following.

Definition 9. *A μ-conservation law for a given PDE $\Delta := F\left(x^i, u^a, u^a_J\right) = 0$ is a $(n-1)$-form π such that $\mathrm{d}^\mu \pi \in \mathcal{C}$ on the solutions to Δ. In particular, this means that $D_i \,\lrcorner\, \mathrm{d}^\mu \pi = 0$ $(i = 1, \ldots, n)$ on the solutions to Δ.*

Now we can prove the extension of Noether's theorem to the case of PDEs.

Theorem 8. *Let X_0 be a divergence variational μ-symmetry for a first-order regular Lagrangian L, and Θ be the Poincaré–Cartan n-form associated with L. Then the following μ-conservation law holds:*

$$D_i \,\lrcorner\, \mathrm{d}^\mu (X_0 \,\lrcorner\, \Theta - \rho) = 0. \tag{5.8}$$

Proof. It is completely analogous to the proof of theorem 7. In particular, on the solution of Euler–Lagrange equations the μ-conservation laws (5.8) explicitly read

$$D_i \left(\frac{\partial L}{\partial u^a_i} \left(\Phi^a - u^a_k \xi^k \right) + \xi^i L - R^i \right) + \Lambda_i \left(\frac{\partial L}{\partial u^a_i} \left(\Phi^a - u^a_k \xi^k \right) + \xi^i L - R^i \right) = 0.$$

□

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